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Mathematics for Economics

Note 3: The Implicit Function Theorem

Note 3 is based on Apostol (1975, Ch. 13), de la Fuente (2000, Ch.5) and Simon and Blume (1994, Ch. 15).

This note discusses the Implicit Function Theorem (IFT). This result plays a key role in economics, particularly in constrained optimization problems and the analysis of comparative statics. The first section develops the IFT for the simplest model of one equation and one exogenous variable. We then extend the analysis to multiple equations and exogenous variables.

## Implicit Function Theorem: One Equation

In general, we are accustom to work with functions of the form $x = f(\alpha)$ where the endogenous variable $x$ is an explicit function of the exogenous variable $\alpha$. This ideal situation does not always occur in economic models. The IFT in its simplest form deals with an equation of the form

$$F(x; \alpha) = 0 \tag{1}$$

where we separate endogenous and exogenous variables by a semicolon.

The problem is to decide whether this equation determines $x$ as a function of $\alpha$. If so, we have $x = x(\alpha)$ for some function $x(.)$, and we say $x(.)$ is defined "implicitly" by (1). Formally, we are interested in two questions

(a) Under which conditions on (1) is $x$ determined as a function of $\alpha$?; and

(b) How do changes in $\alpha$ affect the corresponding value of $x$?

The IFT answers these two questions simultaneously!

The idea behind this fundamental theorem is quite simple. If $F(x; \alpha)$ is a linear function, then the answer to the previous questions is trivial—we elaborate on this point below. If $F(x; \alpha)$ is nonlinear, then the IFT states a set of conditions under which we can use derivatives to construct a linear system that behaves closely to the nonlinear equation around some initial point. We can then address the local behavior of the nonlinear system by studying the properties of the associated linear one. So the results of Notes 1 and 2 turn out to be important here!

Before introducing the IFT let us develop a few examples that clarify the requirements of the theorem and its main implications.

**Example 1.** Let us consider the function $F(x;\alpha) = ax - b\alpha - c$. Here the values that satisfy $F(x;\alpha) = 0$ form a linear equation $ax - b\alpha - c = 0$. Moreover, assuming $a \neq 0$, the values of $x$ and $\alpha$ that satisfy $F(x;\alpha) = 0$ can be expressed as

$$x(\alpha) = \frac{b}{a}\alpha + \frac{c}{a}.$$

If $a \neq 0$, then $x(\alpha)$ is a continuous function of $\alpha$ and $dx(\alpha)/d\alpha = b/a$.

Notice that the partial derivative of $F(x;\alpha)$ with respect to $x$ is $\partial F(x;\alpha)/\partial x = a$. So in this simple case, $x(\alpha)$ exists and is differentiable if and only if $\partial F(x;\alpha)/\partial x \neq 0$. ▲

In Example 1, assuming $\partial F(x;\alpha)/\partial x \neq 0$, $x(\alpha)$ is defined for every initial value of $x$ and $\alpha$. In the next example, $x(\alpha)$ exists only around some specific values of these two variables.

**Example 2.** Let $F(x;\alpha) = x^2 + \alpha^2 - 1$, so that the values of $x$ and $\alpha$ that satisfy $F(x;\alpha) = 0$ form a circle of radius 1 and center $(0,0)$ in $\mathbb{R}^2$.

In this case, for each $\alpha \in (-1,1)$ we have two possible values of $x$ that satisfy $F(x;\alpha) = x^2 + \alpha^2 - 1 = 0$. [Therefore, $x(\alpha)$ is not a function.] Note, however, that if we restrict $x$ to positive values, then we will have the upper half of the circle only, and that does constitute a function, namely, $x(\alpha) = +\sqrt{1-\alpha^2}$.

Similarly, if we restrict $x$ to negative values, then we will have the lower half of the circle only, and that does constitute a function as well, namely, $x(\alpha) = -\sqrt{1-\alpha^2}$.

Here for all $x > 0$ we have that $\partial F(x;\alpha)/\partial x = 2x > 0$; and for all $x < 0$ we have that $\partial F(x;\alpha)/\partial x = 2x < 0$. Then the condition $\partial F(x;\alpha)/\partial x \neq 0$ plays again an important role in the existence and differentiability of $x(\alpha)$. ▲

The last two examples suggest that $\partial F(x;\alpha)/\partial x \neq 0$ is a key ingredient for $x(\alpha)$ to exist, and that in some cases $x(\alpha)$ exists only around some initial values of $x$ and/or $\alpha$. The next example proceeds in a different way, it assumes $x(\alpha)$ exists and studies the behavior of this function with respects to $\alpha$.

**Example 3.** Consider the cubic implicit function

$$F(x;\alpha) = x^3 + \alpha^2 - 3\alpha x - 7 = 0 \tag{2}$$

around the point $x = 3$ and $\alpha = 4$. Suppose that we could find a function $x = x(\alpha)$ that solves (2) around the previous point. Plugging this function in (2) we get

$$[x(\alpha)]^3 + \alpha^2 - 3\alpha x(\alpha) - 7 = 0. \tag{3}$$

Differentiating this expression with respect to $\alpha$ (by using the Chain Rule) we obtain

$$3[x(\alpha)]^2 \frac{dx}{d\alpha}(\alpha) + 2\alpha - 3x(\alpha) - 3\alpha \frac{dx}{d\alpha}(\alpha) = 0. \tag{4}$$

Therefore

$$\frac{dx}{d\alpha}(\alpha) = \frac{1}{3\left\{[x(\alpha)]^2 - \alpha\right\}}[3x(\alpha) - 2\alpha]. \tag{5}$$

At $x = 3$ and $\alpha = 4$ we find

$$\frac{dx}{d\alpha}(\alpha) = \frac{1}{15}. \tag{6}$$

Notice that (5) exists if $[x(\alpha)]^2 - \alpha \neq 0$. Since $\partial F(x;\alpha)/\partial x = 3\left(x^2 - \alpha\right)$, the required condition is again $\partial F(x,\alpha)/\partial x \neq 0$ at the point of interest. ▲

Let us extend Example 3 to a general implicit function $F(x;\alpha) = 0$ around an initial point $(x^*, \alpha^*)$. To this end suppose there is a $C^1$ (continuously differentiable) solution $x = x(\alpha)$ to the equation $F(x;\alpha) = 0$, that is,

$$F[x(\alpha);\alpha] = 0. \tag{7}$$

We can use the Chain Rule to differentiate (7) with respect to $\alpha$ at $\alpha^*$ to obtain

$$\frac{\partial F}{\partial \alpha}[x(\alpha^*);\alpha^*]\frac{d\alpha}{d\alpha} + \frac{\partial F}{\partial x}[x(\alpha^*);\alpha^*]\frac{dx}{d\alpha}(\alpha^*) = 0.$$

Solving for $dx/d\alpha$ we get

$$\frac{dx}{d\alpha}(\alpha^*) = -\frac{\partial F}{\partial \alpha}[x(\alpha^*);\alpha^*] \Big/ \frac{\partial F}{\partial x}[x(\alpha^*);\alpha^*]. \tag{8}$$

The last expression shows that if the solution $x(\alpha)$ to $F(x;\alpha)=0$ exists and is continuously differentiable, then we need $\partial F(x;\alpha)/\partial x \neq 0$ at $[x(\alpha^*);\alpha^*]$ to recover $dx/d\alpha$ at $\alpha^*$. The IFT states that this necessary condition is also a sufficient condition!

**Theorem 1. (Implicit Function Theorem)** Let $F(x;\alpha)$ be a $C^1$ function on a ball about $(x^*,\alpha^*)$ in $\mathbb{R}^2$. Suppose that $F(x^*;\alpha^*)=0$ and consider the expression

$$F(x;\alpha)=0.$$

If $\partial F(x^*;\alpha^*)/\partial x \neq 0$, then there exists a $C^1$ function $x=x(\alpha)$ defined on an open interval $I$ about the point $\alpha^*$ such that:

(a) $F[x(\alpha);\alpha]=0$ for all $\alpha$ in $I$;

(b) $x(\alpha^*)=x^*$; and

(c) $\dfrac{dx}{d\alpha}(\alpha^*)=-\dfrac{\partial F}{\partial\alpha}[x(\alpha^*);\alpha^*]/\dfrac{\partial F}{\partial x}[x(\alpha^*);\alpha^*]$.

**Proof.** See de la Fuente (2000), pp. 207-210. ■

The next example applies the IFT to the standard model of firm behavior in microeconomics. In ECON 501A we will study a general version of this problem. Although in the next example the endogenous variable can in fact be explicitly solved in terms of the exogenous ones, we will use the IFT to state how changes in the latter affect the former. In this way we can corroborate the predictions of the IFT hold.

**Example 4. (Comparative Statics I)** Let us consider a firm that produces a good $y$ by using a single input $x$. The firm sells the output and acquires the input in competitive markets. The market price of $y$ is $p$, and the cost of each unit of $x$ is just $w$. Its technology is given by $f:\mathbb{R}_+\to\mathbb{R}_+$, where $f(x)=x^a$ and $a\in(0,1)$. Its profits are given by

$$\pi(x;p,w)=px^a-wx. \tag{9}$$

The firm selects the input level, $x$, in order to maximize profits. We would like to know how its choice of $x$ is affected by a change in $w$.

Assuming an interior solution, the first-order condition of this optimization problem is

$$\frac{\partial \pi}{\partial x}\left(x^{*}; p, w\right)=p a\left(x^{*}\right)^{a-1}-w=0 \tag{10}$$

for some $x = x^*$. (**Check that the second order condition holds**.)

Notice that here

$$F(x; p, w)=p a\left(x^{*}\right)^{a-1}-w. \tag{11}$$

Since $\partial F\left(x^{*}; p, w\right) / \partial x=p a(a-1)\left(x^{*}\right)^{a-2}<0$, we can use the IFT to get

$$\frac{d x}{d w}(p, w)=-\frac{\partial F}{\partial w}\left[x^{*}; p, w\right] / \frac{\partial F}{\partial x}\left[x^{*}; p, w\right]=-\frac{-1}{p a(a-1)\left(x^{*}\right)^{a-2}}<0. \tag{12}$$

We conclude that if the price of the input increases, then the firm will acquire less of it. That is, the unconditional input demand is decreasing in the input price. [**Confirm this result by finding an explicit expression for $x^*$ in (9) and differentiating it with respect to $w$.**] ▲

The next sub-section extends the previous ideas to a model that involves a systems of equations and many parameters.

## Implicit Function Theorem: System of Equations

The general problem involves a system of several equations and variables. Some of the variables are endogenous, and the other ones are exogenous. The question of interest is under what conditions we can solve these equations for the endogenous variables in terms of the remaining variables (or parameters). The IFT describes these conditions and some conclusions about the effect of the exogenous variables on the endogenous ones. To motivate the requirements of the theorem, we start again with a linear example.

**Example 5.** Consider a linear system of two equations

$$\begin{aligned} a x_{1}+b x_{2}-\alpha_{1}&=0 \\ c x_{1}+d x_{2}-\alpha_{2}&=0 \end{aligned} \quad \text{or} \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\begin{array}{c} x_{1} \\ x_{2} \end{array}\right)=\left(\begin{array}{c} \alpha_{1} \\ \alpha_{2} \end{array}\right). \tag{13}$$

In Note 1 we showed that the linear system (13) has a unique solution for $(x_1, x_2)$ if the determinant of the coefficient matrix is different from 0, i.e. $ad - cb \neq 0$. If this condition holds, we can use Crammer's rule to state

$$
\begin{aligned}
x_1 &= x_1(\alpha_1, \alpha_2) \text{ with } x_1(\alpha_1, \alpha_2) = \left| \begin{pmatrix} \alpha_1 & b \\ \alpha_2 & d \end{pmatrix} \right| / \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = \frac{\alpha_1 d - \alpha_2 b}{ad - cb} \\
x_2 &= x_2(\alpha_1, \alpha_2) \text{ with } x_2(\alpha_1, \alpha_2) = \left| \begin{pmatrix} a & \alpha_1 \\ c & \alpha_2 \end{pmatrix} \right| / \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = \frac{a\alpha_2 - c\alpha_1}{ad - cb}.
\end{aligned}
$$

Then if $ad - cb \neq 0$ we can find a pair of functions $x_1(\alpha_1, \alpha_2)$ and $x_2(\alpha_1, \alpha_2)$ that simultaneously solve (13) for all possible values of the exogenous variables $\alpha_1$ and $\alpha_2$. Moreover, these two functions are $C^1$ and we can differentiate, for instance, $x_1(\alpha_1, \alpha_2)$ with respect to $\alpha_2$ to obtain

$$\frac{\partial x_1(\alpha_1, \alpha_2)}{\partial \alpha_2} = \frac{-b}{ad - cb}.$$

The Jacobian matrix of the left-hand side of system (13) with respect to $(x_1, x_2)$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

The determinant of (14) is $ad - cb$. Therefore, we conclude that the system of equations (13) defines $x_1(\alpha_1, \alpha_2)$ and $x_2(\alpha_1, \alpha_2)$ as continuous functions of the exogenous variables if the determinant of its Jacobian matrix is different from zero. In the general IFT, the nonvanishing condition of the determinant of the Jacobian matrix also plays a fundamental role. This comes about by approximating a system of nonlinear equations by linear ones! ▲

To extend Example 5 to a nonlinear system, let us write the basic system of equations as

$$
\begin{aligned}
F_1(x_1, ..., x_n; \alpha_1, ..., \alpha_m) &= 0 \\
\vdots &= \vdots \\
F_n(x_1, ..., x_n; \alpha_1, ..., \alpha_m) &= 0
\end{aligned} \tag{15}
$$

where $(x_1, ..., x_n)$ is the vector of endogenous variables and $(\alpha_1, ..., \alpha_m)$ are the exogenous ones. Suppose there are $n$ $C^1$ functions in a neighborhood of $(x_1^*, ..., x_n^*; \alpha_1^*, ..., \alpha_m^*)$

$$
\begin{aligned}
x_1 &= x_1(\alpha_1, ..., \alpha_m) \\
\vdots &= \vdots \\
x_n &= x_n(\alpha_1, ..., \alpha_m)
\end{aligned}
$$

that solve the system of equations (15). Then

$$\begin{aligned} F_1\left[x_1\left(\alpha_1,...,\alpha_m\right),...,x_n\left(\alpha_1,...,\alpha_m\right);\alpha_1,...,\alpha_m\right] &= 0 \\ \vdots &= \vdots \\ F_n\left[x_1\left(\alpha_1,...,\alpha_m\right),...,x_n\left(\alpha_1,...,\alpha_m\right);\alpha_1,...,\alpha_m\right] &= 0. \end{aligned} \tag{16}$$

We can use the Chain Rule to differentiate (16) with respect to $\alpha_h$ at $(\alpha_1^*,...,\alpha_m^*)$ to get

$$\begin{aligned} \frac{\partial F_1}{\partial x_1}\frac{\partial x_1}{\partial \alpha_h}+...+\frac{\partial F_1}{\partial x_n}\frac{\partial x_n}{\partial \alpha_h}+\frac{\partial F_1}{\partial \alpha_h} &= 0 \\ \vdots &= \vdots \\ \frac{\partial F_n}{\partial x_1}\frac{\partial x_1}{\partial \alpha_h}+...+\frac{\partial F_n}{\partial x_n}\frac{\partial x_n}{\partial \alpha_h}+\frac{\partial F_n}{\partial \alpha_h} &= 0 \end{aligned} \tag{17}$$

where all the partial derivatives are evaluated at $(x_1^*,...,x_n^*;\alpha_1^*,...,\alpha_m^*)$. This system of equations can be rewritten as

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial \alpha_h} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha_h} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial \alpha_h} \\ \vdots \\ \frac{\partial F_n}{\partial \alpha_h} \end{pmatrix} \tag{18}$$

where the $n \times n$ matrix of the left-hand side is the Jacobian of (16) with respect to $(x_1,...,x_n)$ evaluated at $(x_1^*,...,x_n^*;\alpha_1^*,...,\alpha_m^*)$.

Solving for $(\partial x_1/\partial \alpha_h,...,\partial x_n/\partial \alpha_h)$ we obtain

$$\begin{pmatrix} \frac{\partial x_1}{\partial \alpha_h} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha_h} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial \alpha_h} \\ \vdots \\ \frac{\partial F_n}{\partial \alpha_h} \end{pmatrix}. \tag{19}$$

We see from (19) that if the solution $\left[x_1\left(\alpha_1,...,\alpha_m\right),...,x_n\left(\alpha_1,...,\alpha_m\right)\right]$ to the system of equations (15) exists and is differentiable, then $(\partial x_1/\partial \alpha_h,...,\partial x_n/\partial \alpha_h)$ exists if the determinant of the Jacobian of (15) is different from zero at $(x_1^*,...,x_n^*;\alpha_1^*,...,\alpha_m^*)$. **(Why?)** The IFT shows again that this necessary condition is also a sufficient condition!

**Theorem 2. (Implicit Function Theorem)** Let $F_1,...,F_n : \mathbb{R}^{n+m} \to \mathbb{R}$ be $C^1$ functions.

Consider the system of equations

$$\begin{aligned} F_1(x_1, ..., x_n; \alpha_1, ..., \alpha_m) &= 0 \\ \vdots &= \vdots \\ F_n(x_1, ..., x_n; \alpha_1, ..., \alpha_m) &= 0 \end{aligned} \tag{20}$$

as possibly defining $x_1, ..., x_n$ as implicit functions of $\alpha_1, ..., \alpha_m$. Suppose that $(x_1^*, ..., x_n^*; \alpha_1^*, ..., \alpha_m^*)$ is a solution of (20). If the determinant of the $n \times n$ Jacobian matrix

$$D_{\mathbf{x}}\mathbf{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

evaluated at $(x_1^*, ..., x_n^*; \alpha_1^*, ..., \alpha_m^*)$ is non-zero, then there exist $C^1$ functions

$$\begin{aligned} x_1 &= x_1(\alpha_1, ..., \alpha_m) \\ \vdots &= \vdots \\ x_n &= x_n(\alpha_1, ..., \alpha_m) \end{aligned}$$

defined on an open ball $B$ around $(\alpha_1^*, ..., \alpha_m^*)$ such that

$$\begin{aligned} F_1[x_1(\alpha_1, ..., \alpha_m), ..., x_n(\alpha_1, ..., \alpha_m); \alpha_1, ..., \alpha_m] &= 0 \\ \vdots &= \vdots \\ F_n[x_1(\alpha_1, ..., \alpha_m), ..., x_n(\alpha_1, ..., \alpha_m); \alpha_1, ..., \alpha_m] &= 0 \end{aligned} \tag{21}$$

for all $(\alpha_1, ..., \alpha_m)$ in $B$ and

$$\begin{aligned} x_1^* &= x_1(\alpha_1^*, ..., \alpha_m^*) \\ \vdots &= \vdots \\ x_n^* &= x_n(\alpha_1^*, ..., \alpha_m^*). \end{aligned}$$

Furthermore, we can calculate $\frac{\partial x_1}{\partial \alpha_h}(\alpha_1^*, ..., \alpha_m^*), ..., \frac{\partial x_n}{\partial \alpha_h}(\alpha_1^*, ..., \alpha_m^*)$ as in (19), or solve for one particular element of this vector by using Crammer's rule.

**Proof.** See de la Fuente (2000), pp. 211-212. ■

SB, pp. 360-364, provide an interesting application. We offer a simpler one that extends Example 4 to a firm problem with two inputs.

**Example 6. (Comparative Statics II)** Let us consider again a firm that produces a good $y$, but now let us assume it uses two inputs $x_1$ and $x_2$. The firm sells the output and acquires the inputs in competitive markets. The market price of $y$ is $p$, and the cost of each unit of $x_1$ and $x_2$ is just $w_1$ and $w_2$ respectively. Its technology is given by $f:\mathbb{R}_+^2 \to \mathbb{R}_+$, where $f(x_1,x_2)=x_1^a x_2^b$, $a+b<1$. Its profits take the form

$$\pi(x_1,x_2;p,w_1,w_2)=px_1^a x_2^b - w_1x_1 - w_2x_2. \tag{22}$$

The firm selects $x_1$ and $x_2$ in order to maximize profits. **We aim to know how its choice of $x_1$ is affected by a change in $w_1$.** Notice that now $w_1$ affects the choice of $x_1$ not only in a direct way (as in Example 4) but also indirectly through its effect on the other variable of choice, $x_2$.

Assuming an interior solution, the first-order conditions of this optimization problem are

$$\begin{aligned} \frac{\partial \pi}{\partial x_1}(x_1^*,x_2^*;p,w_1,w_2) &= pa(x_1^*)^{a-1}(x_2^*)^b - w_1 = 0 \\ \frac{\partial \pi}{\partial x_2}(x_1^*,x_2^*;p,w_1,w_2) &= pb(x_1^*)^{a}(x_2^*)^{b-1} - w_2 = 0 \end{aligned} \tag{23}$$

for some $(x_1,x_2)=(x_1^*,x_2^*)$. (As we will study later, the second order conditions hold here by the strict concavity of the production function.)

Let us define

$$\begin{aligned} F_1(x_1^*,x_2^*;p,w_1,w_2) &= pa(x_1^*)^{a-1}(x_2^*)^b - w_1 \\ F_2(x_1^*,x_2^*;p,w_1,w_2) &= pb(x_1^*)^{a}(x_2^*)^{b-1} - w_2. \end{aligned} \tag{24}$$

To achieve our goal we need to differentiate the system of equations (24) with respect to $w_1$

$$\begin{aligned} pa(a-1)(x_1^*)^{a-2}(x_2^*)^b\frac{\partial x_1^*}{\partial w_1} + pab(x_1^*)^{a-1}(x_2^*)^{b-1}\frac{\partial x_2^*}{\partial w_1} - 1 &= 0 \\ pab(x_1^*)^{a-1}(x_2^*)^{b-1}\frac{\partial x_1^*}{\partial w_1} + pb(b-1)(x_1^*)^{a}(x_2^*)^{b-2}\frac{\partial x_2^*}{\partial w_1} - 0 &= 0. \end{aligned} \tag{25}$$

The Jacobian matrix of $\mathbf{F}=(F_1,F_2)^T$ with respect to $\mathbf{x}$ at $\mathbf{x}^*$ is

$$D_{\mathbf{x}}\mathbf{F}(x_1^*,x_2^*)=\begin{pmatrix} pa(a-1)(x_1^*)^{a-2}(x_2^*)^b & pab(x_1^*)^{a-1}(x_2^*)^{b-1} \\ pab(x_1^*)^{a-1}(x_2^*)^{b-1} & pb(b-1)(x_1^*)^{a}(x_2^*)^{b-2} \end{pmatrix}. \tag{26}$$

Notice that in this case $D_{\mathbf{x}}\mathbf{F}(x_1^*, x_2^*; p, w_1, w_2) = pD_{\mathbf{x}}^2 f(x_1^*, x_2^*)$. That is, the Jacobian of $\mathbf{F}$ with respect to $(x_1, x_2)$ is the market price times the Hessian of the production function. A similar structure appears in many other models of microeconomics.

Combining (25) and (26), and rearranging terms, we get

$$\begin{pmatrix} pa(a-1)(x_1^*)^{a-2}(x_2^*)^b & pab(x_1^*)^{a-1}(x_2^*)^{b-1} \\ pab(x_1^*)^{a-1}(x_2^*)^{b-1} & pb(b-1)(x_1^*)^a(x_2^*)^{b-2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{27}$$

The requirement of the IFT is satisfied if the determinant of (26) is non-zero. (**Check that this condition holds**.) By the IFT and Crammer's rule we get

$$\frac{\partial x_1}{\partial w_1}(p, w_1, w_2) = \frac{\left| \begin{pmatrix} 1 & pab(x_1^*)^{a-1}(x_2^*)^{b-1} \\ 0 & pb(b-1)(x_1^*)^a(x_2^*)^{b-2} \end{pmatrix} \right|}{\left| \begin{pmatrix} pa(a-1)(x_1^*)^{a-2}(x_2^*)^b & pab(x_1^*)^{a-1}(x_2^*)^{b-1} \\ pab(x_1^*)^{a-1}(x_2^*)^{b-1} & pb(b-1)(x_1^*)^a(x_2^*)^{b-2} \end{pmatrix} \right|} < 0.$$

We conclude that if the price of input 1 increases, then the firm will acquire less of it. [**Confirm this result by finding an explicit expression for $x_1^*$ and $x_2^*$ in (23) and differentiating $x_1^*$ with respect to $w_1$**.] ▲